

# On matrix inequalities between the power means: counterexamples

Koenraad M.R. Audenaert<sup>1,\*</sup> and Fumio Hiai<sup>2,†</sup>

<sup>1</sup> Department of Mathematics, Royal Holloway, University of London,  
Egham TW20 0EX, United Kingdom

<sup>2</sup> Tohoku University (Emeritus),  
Hakusan 3-8-16-303, Abiko 270-1154, Japan

## Abstract

We prove that the known sufficient conditions on the real parameters  $(p, q)$  for which the matrix power mean inequality  $((A^p + B^p)/2)^{1/p} \leq ((A^q + B^q)/2)^{1/q}$  holds for every pair of matrices  $A, B > 0$  are indeed best possible. The proof proceeds by constructing  $2 \times 2$  counterexamples. The best possible conditions on  $(p, q)$  for which  $\Phi(A^p)^{1/p} \leq \Phi(A^q)^{1/q}$  holds for every unital positive linear map  $\Phi$  and  $A > 0$  are also clarified.

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## 1 Introduction

For each  $n \in \mathbb{N}$  we write  $\mathbb{M}_n$  for the  $n \times n$  complex matrix algebra and  $\mathbb{P}_n$  for the set of positive definite matrices in  $\mathbb{M}_n$ . For each non-zero real parameter  $p$  and for every  $A, B \in \mathbb{P}_n$ , the  $p$ -power mean of  $A, B$  is

$$\left( \frac{A^p + B^p}{2} \right)^{1/p}, \quad (1.1)$$

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\*E-mail: Koenraad.Audenaert@rhul.ac.uk

†E-mail: hiai.fumio@gmail.com

which is also defined for positive invertible operators on an arbitrary Hilbert space. In particular, it is the arithmetic mean when  $p = 1$ , and it is the harmonic mean when  $p = -1$ . Moreover, when  $p = 0$ , it is defined by continuity as

$$\lim_{p \rightarrow 0} \left( \frac{A^p + B^p}{2} \right)^{1/p} = \exp \left( \frac{\log A + \log B}{2} \right), \quad (1.2)$$

which is the so-called *Log-Euclidean mean*, a kind of geometric mean but different from that in the sense of operator means [11]. In fact, (1.1) is not an operator mean except when  $p = \pm 1$ .

In this paper we are concerned with conditions on  $p$  and  $q$  for the validity of the matrix inequality between the power means

$$\left( \frac{A^p + B^p}{2} \right)^{1/p} \leq \left( \frac{A^q + B^q}{2} \right)^{1/q}. \quad (1.3)$$

A more general result involving positive linear maps is known under suitable assumptions on  $p, q$  in [6, 12, 13, 14] (see Theorems 2.1 and 2.2 below). Our interest here is showing that these sufficient conditions of  $p, q$  are best possible for (1.3) to hold. Although the result is naturally expected, no rigorous proof is known to the best of our knowledge. This question for the best possible conditions of  $p, q$  showed up in some concavity/convexity problem of a certain matrix function in [10].

It is useful to write power means in terms of a positive linear map of block-diagonal matrices. Defining a positive linear map  $\Phi : \mathbb{M}_{2n} \rightarrow \mathbb{M}_n$  by

$$\Phi \left( \begin{bmatrix} A & X \\ Y & B \end{bmatrix} \right) := \frac{A + B}{2} \quad (1.4)$$

for matrices in  $\mathbb{M}_{2n}$  partitioned in blocks in  $\mathbb{M}_n$ , one can write for  $A, B \in \mathbb{P}_n$

$$\Phi \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^p \right)^{1/p} = \left( \frac{A^p + B^p}{2} \right)^{1/p}.$$

Therefore, it is also interesting to determine  $p, q$  for which the inequality

$$\Phi(A^p)^{1/p} \leq \Phi(A^q)^{1/q} \quad (1.5)$$

holds for every unital positive linear map  $\Phi$ . Most fundamental in such matrix/operator inequalities are Choi's inequality [4] (extending Davis [5]) and Hansen and Pedersen's Jensen inequality [7].

The paper is organized as follows. In Section 2 we state in more precise terms our problem on the best possible  $p, q$  for matrix inequalities (1.3) and (1.5) together with the known affirmative results. A motivation coming from [10] is also explained. Section 3 is the body of the proof of our main result by constructing counterexamples to (1.3), all of which are given by  $2 \times 2$  matrices. Those are further reformulated to give counterexamples to (1.5) for  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$ .

## 2 Result and motivation

The main aim of this paper is to determine the range of real parameters  $p, q$  for which the matrix inequality between the power means in (1.3) holds. Before stating the main result we first recall the affirmative result, which is known to hold in a more general setting of (1.5). Let  $\mathcal{H}$  and  $\mathcal{K}$  be general Hilbert spaces. Let  $B(\mathcal{H})$  be the algebra of all bounded linear operators on  $\mathcal{H}$  and  $B(\mathcal{H})^{++}$  the set of all positive invertible operators on  $\mathcal{H}$ . Let  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$  be a positive linear map that is unital, i.e.,  $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$ , where  $I_{\mathcal{H}}$  denotes the identity operator on  $\mathcal{H}$ . Then the map

$$A \in B(\mathcal{H})^{++} \longmapsto \Phi(A^p)^{1/p} \in B(\mathcal{K})^{++} \quad (2.1)$$

can be defined for every  $p \in \mathbb{R}$  with  $p \neq 0$ . Indeed, for every  $A \in B(\mathcal{H})^{++}$  and for every  $p \neq 0$ , since  $A^p \geq \delta I_{\mathcal{H}}$  for some  $\delta > 0$ ,  $\Phi(A^p) \geq \delta I_{\mathcal{K}}$  so that  $\Phi(A^p) \in B(\mathcal{K})^{++}$ . Moreover, the following convergence in the operator norm is straightforward:

$$\lim_{p \rightarrow 0} \Phi(A^p)^{1/p} = \exp \Phi(\log A) \quad (2.2)$$

for every  $A \in B(\mathcal{H})^{++}$ . Indeed,

$$\begin{aligned} \frac{1}{p} \log \Phi(A^p) &= \frac{1}{p} \log \Phi(I_{\mathcal{H}} + p \log A + o(p)) = \frac{1}{p} \log(I_{\mathcal{K}} + p \Phi(\log A) + o(p)) \\ &= \Phi(\log A) + o(p), \end{aligned}$$

where  $o(p)$  means that  $o(p)/p \rightarrow 0$  in the operator norm as  $p \rightarrow 0$ . So we shall write  $\Phi(A^p)^{1/p}$  when  $p = 0$  to mean  $\exp \Phi(\log A)$ .

Under the above assumption, we state the following result which can be considered folklore.

**Theorem 2.1.** *Let  $p, q \in \mathbb{R}$ . The operator inequality*

$$\Phi(A^p)^{1/p} \leq \Phi(A^q)^{1/q}$$

*holds for every  $A \in B(\mathcal{H})^{++}$  if  $(p, q)$  satisfies one of the following conditions:*

$$\left\{ \begin{array}{l} p = q, \\ 1 \leq p < q, \\ p < q \leq -1, \\ p \leq -1, q \geq 1, \\ 1/2 \leq p < 1 \leq q, \\ p \leq -1 < q \leq -1/2. \end{array} \right. \quad (2.3)$$

*Proof.* For the convenience of the reader we give a concise proof using Choi's inequality [4, Theorem 2.1]. When  $1 \leq p < q$ , we have  $\Phi(A^p) \leq \Phi(A^q)^{p/q}$  so that  $\Phi(A^p)^{1/p} \leq \Phi(A^q)^{1/q}$ . When  $p \leq -1$  and  $q \geq 1$ , or when  $1/2 \leq p < 1 \leq q$ , we have  $\Phi(A^p)^{1/p} \leq \Phi(A^q)^{1/q}$ . The proof is similar for the remaining cases.  $\square$

Next, let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be Hilbert spaces and  $\Phi_i : B(\mathcal{H}_i) \rightarrow B(\mathcal{K})$  be positive linear maps,  $i = 1, 2$ , such that  $\Phi_1(I_{\mathcal{H}_1}) + \Phi_2(I_{\mathcal{H}_2}) = I_{\mathcal{K}}$ . Define a unital positive linear map  $\Phi : B(\mathcal{H}_1 \oplus \mathcal{H}_2) \rightarrow B(\mathcal{K})$  by

$$\Phi\left(\begin{bmatrix} A & X \\ Y & B \end{bmatrix}\right) := \Phi_1(A) + \Phi_2(B)$$

for  $A \in B(\mathcal{H}_1)$  and  $B \in B(\mathcal{H}_2)$ . For this  $\Phi$ , restricting map (2.1) to  $A \oplus B$  defines

$$(A, B) \in B(\mathcal{H}_1)^{++} \times B(\mathcal{H}_2)^{++} \longmapsto (\Phi_1(A^p) + \Phi_2(B^p))^{1/p} \in B(\mathcal{K})^{++}.$$

When  $p = 0$ , this means  $\exp(\Phi_1(\log A) + \Phi_2(\log B))$  by (2.2). Therefore, the next result is a special case of Theorem 2.1, which was shown in [12, 13, 14] (see also [6, Chapter 4]). In fact, results in more general forms were given there.

**Theorem 2.2.** *Let  $\Phi_i$ ,  $i = 1, 2$ , be as above. Then the operator inequality*

$$(\Phi_1(A^p) + \Phi_2(B^p))^{1/p} \leq (\Phi_1(A^q) + \Phi_2(B^q))^{1/q}$$

*holds for every  $A \in B(\mathcal{H}_1)^{++}$  and  $B \in B(\mathcal{H}_2)^{++}$  if  $(p, q)$  satisfies one of the conditions in (2.3).*

Obviously, when  $\Phi_1(X) = \Phi_2(X) = (1/2)X$  for  $X \in B(\mathcal{H})$ , the expressions in (2.1) and (2.2) reduce to the power mean in (1.1) and the Log-Euclidean mean in (1.2), respectively. Hence, the above theorem says that, in particular, the matrix inequality between the power means in (1.3) holds if  $(p, q)$  satisfies one of (2.3). It is natural to expect that the converse is also true, that is, (2.3) is the optimal range of  $(p, q)$  for which (1.3) holds true. For this converse direction, it seems that no rigorous proof is known so far. Now, the following is our main result, which completely settles the converse direction.

**Theorem 2.3.** *Let  $p, q \in \mathbb{R}$ , and assume that matrix inequality (1.3) holds for every  $A, B \in \mathbb{P}_2$ . Then  $(p, q)$  satisfies one of the conditions in (2.3).*

To prove the theorem, we need to provide counterexamples to (1.3) for any  $(p, q)$  outside the range given in (2.3), which will be done in the next section. It turns out that all counterexamples are  $2 \times 2$  matrices. Restricted to the case  $q = 1$ , the theorem says the well-known fact [8, Proposition 3.1] that the function  $t^p$  on  $(0, \infty)$  is 2-convex if and only if either  $1 \leq p \leq 2$  or  $-1 \leq p \leq 0$ , so 2-convexity implies operator convexity in this case.

Theorem 2.3, with Theorem 2.1, shows that when  $\Phi : \mathbb{M}_4 \rightarrow \mathbb{M}_2$  is (1.4) for  $n = 2$ , matrix inequality (1.5) holds for every  $A \in \mathbb{P}_4$  if and only if  $(p, q)$  satisfies one of (2.3). However, we can reformulate counterexamples in Theorem 2.3 to obtain the following better result. The proof will be given in the last of the next section.

**Theorem 2.4.** *Let  $p, q \in \mathbb{R}$ , and assume that matrix inequality (1.5) holds for every unital completely positive linear map  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$  and every  $A, B \in \mathbb{P}_3$ . Then  $(p, q)$  satisfies one of the conditions in (2.3).*

Related to the above theorem, the following remarks are worth mentioning:

(1) In particular, when  $q = 1$ , the above theorem says that the Jensen inequality  $\Phi(A)^p \leq \Phi(A^p)$  holds for every unital (completely) positive linear map  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$  and every  $A \in \mathbb{P}_3$  if and only if either  $1 \leq p \leq 2$  or  $-1 \leq p \leq 0$ .

(2) Choi [4] gave a convenient counterexample when  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$  is the compression map taking  $A \in \mathbb{M}_3$  to the  $2 \times 2$  top left corner of  $A$ . Choi's example is

$$A := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

for which  $\Phi(A)^4 \not\leq \Phi(A^4)$ . Since this  $A$  is not positive definite, we take

$$B := A + I_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} > 0.$$

Then a numerical computation shows that the signs of the eigenvalues of  $\Phi(B^p) - \Phi(B)^p$  are

$$\begin{cases} -, + & \text{if } p < -1, \\ +, + & \text{if } -1 < p < 0, \\ -, - & \text{if } 0 < p < 1, \\ +, + & \text{if } 1 < p < 2, \\ -, + & \text{if } p > 2. \end{cases}$$

Thus,  $\Phi(B)^p \leq \Phi(B^p)$  holds only if either  $1 \leq p \leq 2$  or  $-1 \leq p \leq 0$ , and it holds reversed only if  $0 \leq p \leq 1$ .

(3) The matrix sizes 3 and 2 in  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$  of Theorem 2.4 are minimal. Indeed, it is well-known that when  $\varphi$  is a positive linear functional on  $\mathbb{M}_n$ , we have  $f(\varphi(A)) \leq \varphi(f(A))$  for every Hermitian  $A \in \mathbb{M}_n$  and every convex function  $f$  defined on an interval containing the eigenvalues of  $A$ . Also, it is known [2, Theorem 2.2] that when  $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_n$  is a unital positive linear map, the inequality  $f(\Phi(A)) \leq \Phi(f(A))$  holds true for every Hermitian  $A \in \mathbb{M}_2$  and every convex function  $f$  as above. Furthermore, we have the next result showing that the situation is also similar for inequality (1.5).

**Theorem 2.5.** *Let  $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_n$  be a unital positive linear map. Then (1.5) holds true for every  $A, B \in \mathbb{P}_2$  and every  $p, q \in \mathbb{R}$  with  $p \leq q$ .*

*Proof.* The proof is similar to that of [2, Theorem 2.2]. Let  $p < q$  be arbitrary and let  $A \in \mathbb{P}_2$ . We may assume by continuity that  $A$  has eigenvalues  $\lambda_1 > \lambda_2$  such that

$\lambda_1 \lambda_2^p \neq \lambda_2 \lambda_1^p$  and  $\lambda_1 \lambda_2^q \neq \lambda_2 \lambda_1^q$ . Then the computation in [2] gives

$$\begin{aligned}\Phi(A^p) &= \frac{\lambda_1^p - \lambda_2^p}{\lambda_1 - \lambda_2} \Phi(A) - \frac{\lambda_2 \lambda_1^p - \lambda_1 \lambda_2^p}{\lambda_1 - \lambda_2}, \\ \Phi(A^q) &= \frac{\lambda_1^q - \lambda_2^q}{\lambda_1 - \lambda_2} \Phi(A) - \frac{\lambda_2 \lambda_1^q - \lambda_1 \lambda_2^q}{\lambda_1 - \lambda_2}.\end{aligned}$$

Since  $\lambda_2 I_n \leq \Phi(A) \leq \lambda_1 I_n$ , the result follows since

$$\left( \frac{\lambda_1^p - \lambda_2^p}{\lambda_1 - \lambda_2} x - \frac{\lambda_2 \lambda_1^p - \lambda_1 \lambda_2^p}{\lambda_1 - \lambda_2} \right)^{1/p} \leq \left( \frac{\lambda_1^q - \lambda_2^q}{\lambda_1 - \lambda_2} x - \frac{\lambda_2 \lambda_1^q - \lambda_1 \lambda_2^q}{\lambda_1 - \lambda_2} \right)^{1/q},$$

that is,

$$\left( \frac{x - \lambda_2}{\lambda_1 - \lambda_2} \lambda_1^p + \frac{\lambda_1 - x}{\lambda_1 - \lambda_2} \lambda_2^p \right)^{1/p} \leq \left( \frac{x - \lambda_2}{\lambda_1 - \lambda_2} \lambda_1^q + \frac{\lambda_1 - x}{\lambda_1 - \lambda_2} \lambda_2^q \right)^{1/q}$$

for any  $x \in [\lambda_2, \lambda_1]$ . □

In the rest of the section we explain what motivated us to prove the optimality of conditions (2.3) for the validity of (1.3). In [10] we discussed joint concavity/convexity of the trace function

$$(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \longmapsto \text{Tr} \left\{ \Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2} \right\}^s,$$

where  $p, q, s$  are real parameters,  $n, m, l \in \mathbb{N}$ , and  $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_l$  and  $\Psi : \mathbb{M}_m \rightarrow \mathbb{M}_l$  are (strictly) positive linear maps. We are interested in extending concavity/convexity results under trace to those under symmetric (anti-) norms. (The notion of symmetric anti-norms was introduced in [3].) For instance, we are interested in joint convexity of the norm function

$$(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \longmapsto \left\| \left\{ \Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2} \right\}^s \right\|,$$

where  $\|\cdot\|$  is a symmetric norm on  $\mathbb{M}_l$ . This joint convexity for any symmetric norm can be reduced to that for the Ky Fan  $k$ -norms for  $k = 1, \dots, l$ . Although the problem for all Ky Fan norms seems difficult, we could settle in [10] the special case where  $k = 1$ , i.e.,  $\|\cdot\|$  is the operator norm  $\|\cdot\|_\infty$  (another special case where  $k = l$  is the original situation under trace). In [10] we proved

**Theorem 2.6.** *Under the above assumption, the function*

$$(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \longmapsto \left\| \left\{ \Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2} \right\}^s \right\|_\infty$$

*is jointly convex if one of the following six conditions is satisfied:*

$$\begin{cases} -1 \leq p, q \leq 0 \text{ and } s > 0, \\ -1 \leq p \leq 0, \ 1 \leq q \leq 2, \ p + q > 0 \text{ and } s \geq 1/(p + q), \\ 1 \leq p \leq 2, \ -1 \leq q \leq 0, \ p + q > 0 \text{ and } s \geq 1/(p + q), \end{cases} \quad (2.4)$$

*and their counterparts where  $(p, q, s)$  is replaced with  $(-p, -q, -s)$ .*

Moreover, for the optimality of the above conditions in (2.4) for  $(p, q, s)$  we proved

**Theorem 2.7.** *The function*

$$(A, B) \in \mathbb{P}_n \times \mathbb{P}_n \longmapsto \|(A^{p/2} B^q A^{p/2})^s\|_\infty \quad (2.5)$$

*is jointly convex for every  $n \in \mathbb{N}$  (or equivalently, for fixed  $n = 2$ ) if and only if  $(p, q, s)$  satisfies one of the conditions in (2.4) and their counterparts of  $(-p, -q, -s)$  in place of  $(p, q, s)$ .*

The “if” part of Theorem 2.7 is an obvious special case of Theorem 2.6. To prove the “only if” part, we observed that, for each  $n \in \mathbb{N}$ ,  $p, q \neq 0$  and  $s > 0$ , if (2.5) is jointly convex then

$$\left( \frac{A^{1/q} + B^{1/q}}{2} \right)^q \leq \left( \frac{A^{-1/p} + B^{-1/p}}{2} \right)^{-p}$$

holds for every  $A, B \in \mathbb{P}_n$ . In this way, the matrix inequality between the power means shows up, and the restriction on  $(p, q)$  obtained in Theorem 2.3 is crucial to prove Theorem 2.7. So we need to prove Theorem 2.3 to complete the proof of Theorem 2.7 in [10], which is our main motivation here, though Theorem 2.3 is certainly of independent interest.

### 3 Counterexamples

This section is mostly devoted to the proof of Theorem 2.3 by constructing counterexamples. It is obvious that the condition  $p \leq q$  is necessary for (1.3) to hold for the numerical function (i.e., for  $A = aI$  and  $B = bI$  with  $a, b \in (0, \infty)$ ). From the obvious identities

$$\begin{aligned} \left( \frac{A^p + B^p}{2} \right)^{1/p} &= \left\{ \left( \frac{(A^{-1})^{-p} + (B^{-1})^{-p}}{2} \right)^{-1/p} \right\}^{-1}, \quad p \neq 0, \\ \exp\left(\frac{\log A + \log B}{2}\right) &= \left\{ \exp\left(\frac{\log A^{-1} + \log B^{-1}}{2}\right) \right\}^{-1}, \end{aligned}$$

it is also obvious that, for each  $n \in \mathbb{N}$ , (1.3) holds for every  $A, B \in \mathbb{P}_n$  if and only if (1.3) with  $(-q, -p)$  in place of  $(p, q)$  holds for every  $A, B \in \mathbb{P}_n$ . Therefore, it suffices to provide counterexamples for any  $(p, q)$  such that either  $-1 < p < 1/2$  and  $q > \max\{0, p\}$ , or  $1/2 \leq p < q < 1$ . Below we divide our job into three cases which cover all of such  $(p, q)$ .

#### 3.1 Case $-1 < p < 1/2$ , $p \neq 0$ and $q > \max\{0, p\}$

For each  $x, y > 0$  and  $\theta \in \mathbb{R}$  define  $A, B_\theta \in \mathbb{P}_2$  by

$$A := \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, \quad B_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

**Lemma 3.1.** *Let  $p, q \in \mathbb{R} \setminus \{0\}$  and  $x, y > 0$  be such that  $x^p + y^p \neq 2$ ,  $x^q + y^q \neq 2$  and  $((x^p + y^p)/2)^{1/p} \neq ((x^q + y^q)/2)^{1/q}$ . Then we have*

$$\begin{aligned} & \det \left\{ \left( \frac{A^q + B_\theta^q}{2} \right)^{1/q} - \left( \frac{A^p + B_\theta^p}{2} \right)^{1/p} \right\} \\ &= \theta^2 \left[ \frac{1}{2} \left\{ \frac{(1-x^p)(1-y^p)}{p(2-x^p-y^p)} - \frac{(1-x^q)(1-y^q)}{q(2-x^q-y^q)} \right\} \left\{ \left( \frac{x^q + y^q}{2} \right)^{1/q} - \left( \frac{x^p + y^p}{2} \right)^{1/p} \right\} \right. \\ & \quad \left. - \left\{ \frac{1-y^p}{2-x^p-y^p} - \frac{1-y^q}{2-x^q-y^q} \right\}^2 \left\{ 1 - \left( \frac{x^p + y^p}{2} \right)^{1/p} \right\} \left\{ 1 - \left( \frac{x^q + y^q}{2} \right)^{1/q} \right\} \right] \\ & \quad + o(\theta^2) \quad \text{as } \theta \rightarrow 0. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} A^p + B_\theta^p &= \begin{bmatrix} 2 - (1-y^p) \sin^2 \theta & (1-y^p) \frac{\sin 2\theta}{2} \\ (1-y^p) \frac{\sin 2\theta}{2} & x^p + y^p + (1-y^p) \sin^2 \theta \end{bmatrix} \\ &= G + \theta H + \theta^2 K + o(\theta^2), \end{aligned}$$

where

$$G := \begin{bmatrix} 2 & 0 \\ 0 & x^p + y^p \end{bmatrix}, \quad H := \begin{bmatrix} 0 & 1-y^p \\ 1-y^p & 0 \end{bmatrix}, \quad K := \begin{bmatrix} -(1-y^p) & 0 \\ 0 & 1-y^p \end{bmatrix}.$$

We apply the Taylor formula with Fréchet derivatives (see e.g., [9, Theorem 2.3.1]) to obtain

$$(A^p + B_\theta^p)^{1/p} = G^{1/p} + D(x^{1/p})(G)(\theta H + \theta^2 K) + \frac{1}{2} D^2(x^{1/p})(G)(\theta H, \theta H) + o(\theta^2),$$

where the second and the third terms in the right-hand side are the first and the second Fréchet derivatives of  $X \in \mathbb{P}_2 \mapsto X^{1/p} \in \mathbb{P}_2$  at  $G$ , respectively. By Daleckii and Krein's derivative formula (see [1, Theorem V.3.3], [9, Theorem 2.3.1]) we have

$$\begin{aligned} & D(x^{1/p})(G)(\theta H + \theta^2 K) \\ &= \begin{bmatrix} (x^{1/p})^{[1]}(2, 2) & (x^{1/p})^{[1]}(2, x^p + y^p) \\ (x^{1/p})^{[1]}(2, x^p + y^p) & (x^{1/p})^{[1]}(x^p + y^p, x^p + y^p) \end{bmatrix} \circ (\theta H + \theta^2 K) \\ &= \begin{bmatrix} \frac{1}{p} 2^{\frac{1}{p}-1} & \frac{2^{1/p-(x^p+y^p)^{1/p}}}{2-x^p-y^p} \\ \frac{2^{1/p-(x^p+y^p)^{1/p}}}{2-x^p-y^p} & \frac{1}{p} (x^p + y^p)^{\frac{1}{p}-1} \end{bmatrix} \circ (\theta H + \theta^2 K) \\ &= \theta \begin{bmatrix} 0 & \frac{2^{1/p-(x^p+y^p)^{1/p}}(1-y^p)}{2-x^p-y^p} \\ \frac{2^{1/p-(x^p+y^p)^{1/p}}(1-y^p)}{2-x^p-y^p} & 0 \end{bmatrix} \\ & \quad + \theta^2 \begin{bmatrix} -\frac{1}{p} 2^{\frac{1}{p}-1} (1-y^p) & 0 \\ 0 & \frac{1}{p} (x^p + y^p)^{\frac{1}{p}-1} (1-y^p) \end{bmatrix}, \end{aligned}$$



where  $(x^{1/p})^{[1]}$  denotes the first divided difference of  $x^{1/p}$  and  $\circ$  means the Schur (or Hadamard) product. For the second divided difference of  $x^{1/p}$  we compute

$$\begin{aligned}(x^{1/p})^{[2]}(2, 2, x^p + y^p) &= \frac{(\frac{1}{p} - 1)2^{1/p} - \frac{1}{p}2^{\frac{1}{p}-1}(x^p + y^p) + (x^p + y^p)^{1/p}}{(2 - x^p - y^p)^2}, \\ (x^{1/p})^{[2]}(2, x^p + y^p, x^p + y^p) &= \frac{2^{1/p} - \frac{2}{p}(x^p + y^p)^{\frac{1}{p}-1} + (\frac{1}{p} - 1)(x^p + y^p)^{1/p}}{(2 - x^p - y^p)^2},\end{aligned}$$

and hence we have

$$\begin{aligned}& \frac{1}{2}D^2(x^{1/2})(G)(\theta H, \theta H) \\ &= \theta^2 \begin{bmatrix} \frac{(\frac{1}{p}-1)2^{1/p} - \frac{1}{p}2^{\frac{1}{p}-1}(x^p+y^p) + (x^p+y^p)^{1/p}}{(2-x^p-y^p)^2}(1-y^p)^2 & 0 \\ 0 & \frac{2^{1/p} - \frac{2}{p}(x^p+y^p)^{\frac{1}{p}-1} + (\frac{1}{p}-1)(x^p+y^p)^{1/p}}{(2-x^p-y^p)^2}(1-y^p)^2 \end{bmatrix}.\end{aligned}$$

(In the above computation we have used the assumption that  $x^p + y^p \neq 2$ .) Therefore, it follows that

$$\left(\frac{A^p + B_\theta^p}{2}\right)^{1/p} = \begin{bmatrix} 1 + \alpha_p^{(1,1)}\theta^2 & \alpha_p^{(1,2)}\theta \\ \alpha_p^{(1,2)}\theta & \left(\frac{x^p + y^p}{2}\right)^{1/p} + \alpha_p^{(2,2)}\theta^2 \end{bmatrix} + o(\theta^2), \quad (3.1)$$

where

$$\begin{aligned}\alpha_p^{(1,1)} &:= -\frac{1}{2p}(1 - y^p) + \frac{(2 - 2p) - (x^p + y^p) + 2p2^{-1/p}(x^p + y^p)^{1/p}}{2p(2 - x^p - y^p)^2}(1 - y^p)^2 \\ &= -\frac{1}{2p}(1 - y^p) + \frac{(1 - y^p)^2}{2p(2 - x^p - y^p)} - \frac{2^{-1/p}(1 - y^p)^2\{2^{1/p} - (x^p + y^p)^{1/p}\}}{(2 - x^p - y^p)^2} \\ &= -\frac{(1 - x^p)(1 - y^p)}{2p(2 - x^p - y^p)} - \frac{(1 - y^p)^2\{1 - (\frac{x^p + y^p}{2})^{1/p}\}}{(2 - x^p - y^p)^2}, \\ \alpha_p^{(1,2)} &:= \frac{(1 - y^p)\{1 - (\frac{x^p + y^p}{2})^{1/p}\}}{(2 - x^p - y^p)}.\end{aligned}$$

(The form of  $\alpha_p^{(2,2)}$  is not written down here since it is unnecessary in the computation below.) By assumption  $((x^p + y^p)/2)^{1/p} \neq ((x^q + y^q)/2)^{1/q}$ , we arrive at

$$\begin{aligned}& \det\left\{\left(\frac{A^q + B_\theta^q}{2}\right)^{1/q} - \left(\frac{A^p + B_\theta^p}{2}\right)^{1/p}\right\} \\ &= \theta^2 \left[ \{\alpha_q^{(1,1)} - \alpha_p^{(1,1)}\} \left\{\left(\frac{x^q + y^q}{2}\right)^{1/q} - \left(\frac{x^p + y^p}{2}\right)^{1/p}\right\} - \{\alpha_q^{(1,2)} - \alpha_p^{(1,2)}\}^2 \right] + o(\theta^2).\end{aligned}$$

The above formula inside the big bracket is equal to the sum of the following  $\Delta_1$  and  $\Delta_2$ :

$$\Delta_1 := \frac{1}{2} \left\{ \frac{(1 - x^p)(1 - y^p)}{p(2 - x^p - y^p)} - \frac{(1 - x^q)(1 - y^q)}{q(2 - x^q - y^q)} \right\} \left\{ \left(\frac{x^q + y^q}{2}\right)^{1/q} - \left(\frac{x^p + y^p}{2}\right)^{1/p} \right\},$$

$$\Delta_2 := \left\{ -\frac{(1-y^q)^2(1-(\frac{x^q+y^q}{2})^{1/q})}{(2-x^q-y^q)^2} + \frac{(1-y^p)^2(1-(\frac{x^p+y^p}{2})^{1/p})}{(2-x^p-y^p)^2} \right\} \\ \times \left\{ \left( \frac{x^q+y^q}{2} \right)^{1/q} - \left( \frac{x^p+y^p}{2} \right)^{1/p} \right\} \\ - \left\{ \frac{(1-y^q)(1-(\frac{x^q+y^q}{2})^{1/q})}{(2-x^q-y^q)} - \frac{(1-y^p)(1-(\frac{x^p+y^p}{2})^{1/p})}{(2-x^p-y^p)} \right\}^2.$$

Letting  $w_p := 1 - ((x^p + y^p)/2)^{1/p}$  we furthermore compute

$$\Delta_2 = \left\{ -\frac{(1-y^q)^2 w_q}{(2-x^q-y^q)^2} + \frac{(1-y^p)^2 w_p}{(2-x^p-y^p)^2} \right\} (-w_q + w_p) \\ - \left\{ \frac{(1-y^q) w_q}{(2-x^q-y^q)} - \frac{(1-y^p) w_p}{(2-x^p-y^p)} \right\}^2 \\ = \left\{ \frac{1-y^p}{2-x^p-y^p} - \frac{1-y^q}{2-x^q-y^q} \right\}^2 w_p w_q,$$

and the lemma follows from the above expressions of  $\Delta_1$  and  $\Delta_2$ .  $\square$

Now, let  $-1 < p < 1/2$ ,  $p \neq 0$  and  $q > \max\{0, p\}$ . We prove that

$$\left( \frac{A^p + B_\theta^p}{2} \right)^{1/p} \not\leq \left( \frac{A^q + B_\theta^q}{2} \right)^{1/q}$$

for some  $x, y > 0$  and some  $\theta > 0$ . Suppose on the contrary that

$$\left( \frac{A^p + B_\theta^p}{2} \right)^{1/p} \leq \left( \frac{A^q + B_\theta^q}{2} \right)^{1/q}$$

for all  $x, y > 0$  and all  $\theta > 0$ . Let  $0 < x < 1$  and  $y = x^2$ . Then it is clear that

$$x^p + x^{2p} \neq 2, \quad x^q + x^{2q} < 2,$$

$$\left( \frac{x^p + x^{2p}}{2} \right)^{1/p} = x \left( \frac{1 + x^p}{2} \right)^{1/p} < x \left( \frac{1 + x^q}{2} \right)^{1/q} = \left( \frac{x^q + x^{2q}}{2} \right)^{1/q}.$$

Hence, by Lemma 3.1 we must have

$$\frac{1}{2} \left\{ \frac{(1-x^p)(1-x^{2p})}{p(2-x^p-x^{2p})} - \frac{(1-x^q)(1-x^{2q})}{q(2-x^q-x^{2q})} \right\} \left\{ \left( \frac{x^q + x^{2q}}{2} \right)^{1/q} - \left( \frac{x^p + x^{2p}}{2} \right)^{1/p} \right\} \\ - \left\{ \frac{1-x^{2p}}{2-x^p-x^{2p}} - \frac{1-x^{2q}}{2-x^q-x^{2q}} \right\}^2 \left\{ 1 - \left( \frac{x^p + x^{2p}}{2} \right)^{1/p} \right\} \left\{ 1 - \left( \frac{x^q + x^{2q}}{2} \right)^{1/q} \right\} \\ \geq 0. \quad (3.2)$$

When  $0 < p < 1/2$  and  $q > p$ , we have as  $x \searrow 0$

$$\left(\frac{x^q + x^{2q}}{2}\right)^{1/q} - \left(\frac{x^p + x^{2p}}{2}\right)^{1/p} = \frac{x(1+x^q)^{1/q}}{2^{1/q}} - \frac{x(1+x^p)^{1/p}}{2^{1/p}} \approx \frac{2^{1/p} - 2^{1/q}}{2^{\frac{1}{p} + \frac{1}{q}}} x \quad (3.3)$$

and

$$\frac{1 - x^{2p}}{2 - x^p - x^{2p}} - \frac{1 - x^{2q}}{2 - x^q - x^{2q}} = \frac{x^p - x^q - x^{2p} + x^{2q} - x^{p+2q} + x^{2p+q}}{(2 - x^p - x^{2p})(2 - x^q - x^{2q})} \approx \frac{x^p}{4}.$$

Therefore, the dominant term of the left-hand side of (3.2) is

$$\frac{2^{1/p} - 2^{1/q}}{2^{1+\frac{1}{p}+\frac{1}{q}}} \left( \frac{1}{2p} - \frac{1}{2q} \right) x - \frac{x^{2p}}{16} < 0$$

thanks to  $2p < 1$  when  $x > 0$  is sufficiently small. This contradicts (3.2).

When  $-1 < p < 0$  and  $q > 0$ , we have the same estimation (3.3), and moreover

$$\frac{(1-x^p)(1-x^{2p})}{p(2-x^p-x^{2p})} - \frac{(1-x^q)(1-x^{2q})}{q(2-x^q-x^{2q})} \approx -\frac{x^p}{p}$$

and

$$\frac{1 - x^{2p}}{2 - x^p - x^{2p}} - \frac{1 - x^{2q}}{2 - x^q - x^{2q}} \approx 1 - \frac{1}{2} = \frac{1}{2} \quad \text{as } x \searrow 0.$$

Therefore, the left-hand side of (3.2) is dominantly

$$\frac{2^{1/p} - 2^{1/q}}{2^{1+\frac{1}{p}+\frac{1}{q}}} \left( -\frac{x^{p+1}}{p} \right) - \frac{1}{4} < 0$$

thanks to  $p+1 > 0$  for  $x > 0$  sufficiently small, and we have a contradiction again.

### 3.2 Case $p = 0 < q$

For  $x, y > 0$  let  $A, B_\theta \in \mathbb{P}_2$  be the same as in Section 3.1. The following is the counterpart of Lemma 3.1 in the case  $p = 0$ . The expression here can easily be obtained by taking the limit of that in Lemma 3.1 as  $p \rightarrow 0$ . However, deriving the expression in this way is not a rigorous proof, so we sketch an independent proof.

**Lemma 3.2.** *Let  $q \in \mathbb{R} \setminus \{0\}$  and  $x, y > 0$  be such that  $xy \neq 1$ ,  $x^q + y^q \neq 2$  and  $x \neq y$  (hence  $\sqrt{xy} \neq ((x^q + y^q)/2)^{1/q}$ ). Then we have*

$$\begin{aligned} & \det \left\{ \left( \frac{A^q + B_\theta^q}{2} \right)^{1/q} - \exp \left( \frac{\log A + \log B_\theta}{2} \right) \right\} \\ &= \theta^2 \left[ -\frac{1}{2} \left\{ \frac{\log x \cdot \log y}{\log xy} + \frac{(1-x^q)(1-y^q)}{q(2-x^q-y^q)} \right\} \left\{ \left( \frac{x^q + y^q}{2} \right)^{1/q} - \sqrt{xy} \right\} \right. \\ & \quad \left. - \left\{ \frac{\log y}{\log xy} - \frac{1-y^q}{2-x^q-y^q} \right\}^2 (1-\sqrt{xy}) \left\{ 1 - \left( \frac{x^q + y^q}{2} \right)^{1/q} \right\} \right] \\ & \quad + o(\theta^2) \quad \text{as } \theta \rightarrow 0. \end{aligned}$$

*Proof.* We have

$$\begin{aligned}\log A + \log B_\theta &= \begin{bmatrix} \log y \cdot \sin^2 \theta & -\log y \cdot \frac{\sin 2\theta}{2} \\ -\log y \cdot \frac{\sin 2\theta}{2} & \log xy - \log y \cdot \sin^2 \theta \end{bmatrix} \\ &= G + \theta H + \theta^2 K + o(\theta^2),\end{aligned}$$

where

$$G := \begin{bmatrix} 0 & 0 \\ 0 & \log xy \end{bmatrix}, \quad H := \begin{bmatrix} 0 & -\log y \\ -\log y & 0 \end{bmatrix}, \quad K := \begin{bmatrix} \log y & 0 \\ 0 & -\log y \end{bmatrix}.$$

As in the proof of Lemma 3.1,

$$\begin{aligned}\exp\left(\frac{\log A + \log B_\theta}{2}\right) &= e^{G/2} + D(e^x)(G/2)\left(\theta \frac{H}{2} + \theta^2 \frac{K}{2}\right) + \frac{1}{2}D^2(e^x)(G/2)\left(\theta \frac{H}{2}, \theta \frac{H}{2}\right) + o(\theta^2), \\ D(e^x)(G/2)\left(\theta \frac{H}{2} + \theta^2 \frac{K}{2}\right) &= \theta \begin{bmatrix} 0 & \frac{(1-\sqrt{xy})\log y}{\log xy} \\ \frac{(1-\sqrt{xy})\log y}{\log xy} & 0 \end{bmatrix} + \theta^2 \begin{bmatrix} \frac{\log y}{2} & 0 \\ 0 & -\frac{\sqrt{xy}\log y}{2} \end{bmatrix}, \\ \frac{1}{2}D^2(e^x)(G/2)\left(\theta \frac{H}{2}, \theta \frac{H}{2}\right) &= \theta^2 \begin{bmatrix} -\frac{\log^2 y}{2\log xy} - \frac{(1-\sqrt{xy})\log^2 y}{\log^2 xy} & 0 \\ 0 & \frac{\sqrt{xy}\log^2 y}{2\log xy} + \frac{(1-\sqrt{xy})\log^2 y}{\log^2 xy} \end{bmatrix},\end{aligned}$$

where we have used assumption  $xy \neq 1$ . Therefore, we write

$$\exp\left(\frac{\log A + \log B_\theta}{2}\right) = \begin{bmatrix} 1 + \alpha_0^{(1,1)}\theta^2 & \alpha_0^{(1,2)}\theta \\ \alpha_0^{(1,2)}\theta & \sqrt{xy} + \alpha_0^{(2,2)}\theta^2 \end{bmatrix} + o(\theta^2), \quad (3.4)$$

where

$$\begin{aligned}\alpha_0^{(1,1)} &:= \frac{\log y}{2} - \frac{\log^2 y}{2\log xy} - \frac{(1-\sqrt{xy})\log^2 y}{\log^2 xy} = \frac{\log x \cdot \log y}{2\log xy} - \frac{(1-\sqrt{xy})\log^2 y}{\log^2 xy}, \\ \alpha_0^{(1,2)} &:= \frac{(1-\sqrt{xy})\log y}{\log xy}.\end{aligned}$$

Since  $\sqrt{xy} \neq ((x^q + y^q)/2)^{1/q}$  by assumption, we obtain, by (3.4) and (3.1) with  $q$ ,

$$\begin{aligned}\det\left\{\left(\frac{A^q + B_\theta^q}{2}\right)^{1/q} - \exp\left(\frac{\log A + \log B_\theta}{2}\right)\right\} &= \theta^2 \left[ \left\{ \alpha_q^{(1,1)} - \alpha_0^{(1,1)} \right\} \left\{ \left(\frac{x^q + y^q}{2}\right)^{1/q} - \sqrt{xy} \right\} - \left\{ \alpha_q^{(1,2)} - \alpha_0^{(1,2)} \right\}^2 \right] + o(\theta^2).\end{aligned}$$

Letting  $w_0 := 1 - \sqrt{xy}$  as well as  $w_q := 1 - ((x^q + y^q)/2)^{1/q}$  we compute the expression in the above big bracket as

$$\left\{ -\frac{(1-x^q)(1-y^q)}{2q(2-x^q-y^q)} - \frac{\log x \cdot \log y}{2\log xy} \right\} \left\{ \left(\frac{x^q + y^q}{2}\right)^{1/q} - \sqrt{xy} \right\}$$

$$\begin{aligned}
& + \left\{ -\frac{(1-y^q)w_q}{2-x^q-y^q} + \frac{w_0 \log^2 y}{\log^2 xy} \right\} (-w_q + w_0) - \left\{ \frac{(1-y^q)^2 w_q}{(2-x^q-y^q)^2} - \frac{w_0 \log y}{\log xy} \right\}^2 \\
& = -\frac{1}{2} \left\{ \frac{(1-x^q)(1-y^q)}{2q(2-x^q-y^q)} + \frac{\log x \cdot \log y}{2 \log xy} \right\} \left\{ \left( \frac{x^q + y^q}{2} \right)^{1/q} - \sqrt{xy} \right\} \\
& \quad - \left\{ \frac{\log y}{\log xy} - \frac{1-y^q}{2-x^q-y^q} \right\}^2 w_0 w_q,
\end{aligned}$$

and the assertion follows.  $\square$

Now, let  $q > 0$ . We suppose that

$$\exp\left(\frac{\log A + \log B_\theta}{2}\right) \leq \left(\frac{A^q + B_\theta^q}{2}\right)^{1/q}$$

for all  $x, y > 0$  and all  $\theta > 0$ . Let  $0 < x < 1$  and  $y = x^2$ , so  $x^q + y^q \neq 2$  and  $x \neq y$ . Hence, by Lemma 3.2 we must have

$$\begin{aligned}
& -\frac{1}{2} \left\{ \frac{2}{3} \log x + \frac{(1-x^q)(1-x^{2q})}{q(2-x^q-x^{2q})} \right\} \left\{ \left( \frac{x^q + x^{2q}}{2} \right)^{1/q} - x^{3/2} \right\} \\
& \quad - \left\{ \frac{2}{3} - \frac{1-x^{2q}}{2-x^q-x^{2q}} \right\}^2 (1-x^{3/2}) \left\{ 1 - \left( \frac{x^q + x^{2q}}{2} \right)^{1/q} \right\} \geq 0. \quad (3.5)
\end{aligned}$$

As  $x \searrow 0$  we have

$$\left( \frac{x^q + x^{2q}}{2} \right)^{1/q} - x^{3/2} \approx \frac{1}{2^{1/q}} x$$

so that the left-hand side of (3.5) is dominantly

$$-\frac{1}{3 \cdot 2^{1/q}} x \log x - \left( \frac{2}{3} - \frac{1}{2} \right)^2 = -\frac{1}{3 \cdot 2^{1/q}} x \log x - \frac{1}{36} < 0,$$

a contradiction. Hence it has been shown that, for every  $q > 0$ ,

$$\exp\left(\frac{\log A + \log B_\theta}{2}\right) \not\leq \left(\frac{A^q + B_\theta^q}{2}\right)^{1/q}$$

for some  $x, y > 0$  and some  $\theta > 0$ .

### 3.3 Case $0 < p < q < 1$

For  $\theta \in \mathbb{R}$  define  $2 \times 2$  positive semidefinite matrices

$$A := \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_\theta := \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}.$$

Indeed, the latter is  $B_\theta$  in Section 3.1 with  $y = 0$  while the former is slightly different from  $A$  in Section 3.1 with  $x = 0$ .

**Lemma 3.3.** For every  $p, q \in (0, 1)$ ,

$$\begin{aligned} & \det \left\{ \left( \frac{A^q + B_\theta^q}{2} \right)^{1/q} - \left( \frac{A^p + B_\theta^p}{2} \right)^{1/p} \right\} \\ &= -\theta^2 \left( \frac{2^p + 1}{2} \right)^{1/p} \left( \frac{2^q + 1}{2} \right)^{1/q} \left( \frac{1}{2^p + 1} - \frac{1}{2^q + 1} \right)^2 + o(\theta^2) \quad \text{as } \theta \rightarrow 0. \end{aligned}$$

*Proof.* Since  $(A^p + B_\theta^p)/2$  is singular at  $\theta = 0$  and  $x^{1/p}$  is singular at  $x = 0$ , the Taylor formula applied in Sections 3.1 and 3.2 cannot be used. However, a direct approximate computation is not difficult as below. Since  $B$  is a rank one projection, we write

$$\frac{A^p + B_\theta^p}{2} = \begin{bmatrix} \frac{2^p + 1 - \sin^2 \theta}{2} & \frac{\sin 2\theta}{4} \\ \frac{\sin 2\theta}{4} & \frac{\sin^2 \theta}{2} \end{bmatrix} = \frac{2^p + 1}{4} \begin{bmatrix} 1 + a & b \\ b & 1 - a \end{bmatrix},$$

where

$$a := 1 - \frac{2 \sin^2 \theta}{2^p + 1}, \quad b := \frac{\sin 2\theta}{2^p + 1}.$$

Observe that  $\begin{bmatrix} 1 + a & b \\ b & 1 - a \end{bmatrix}$  has the eigenvalues  $1 + c$  and  $1 - c$  with  $c := \sqrt{a^2 + b^2}$  ( $< 1$ ) and the eigenvectors are  $\begin{bmatrix} c + a \\ b \end{bmatrix}$  and  $\begin{bmatrix} c - a \\ -b \end{bmatrix}$ , respectively, from which one can compute

$$\begin{aligned} & \left( \frac{A^p + B_\theta^p}{2} \right)^{1/p} \\ &= \left( \frac{2^p + 1}{4} \right)^{1/p} \begin{bmatrix} c + a & c - a \\ b & -b \end{bmatrix} \begin{bmatrix} (1 + c)^{1/p} & 0 \\ 0 & (1 - c)^{1/p} \end{bmatrix} \begin{bmatrix} c + a & c - a \\ b & -b \end{bmatrix}^{-1} \\ &= \left( \frac{2^p + 1}{4} \right)^{1/p} \begin{bmatrix} \frac{(1+c)^{1/p} + (1-c)^{1/p}}{2} + \frac{(1+c)^{1/p} - (1-c)^{1/p}}{2c} a & \frac{(1+c)^{1/p} - (1-c)^{1/p}}{2c} b \\ \frac{(1+c)^{1/p} - (1-c)^{1/p}}{2c} b & \frac{(1+c)^{1/p} + (1-c)^{1/p}}{2} - \frac{(1+c)^{1/p} - (1-c)^{1/p}}{2c} a \end{bmatrix}. \end{aligned}$$

As  $\theta \searrow 0$  we compute

$$a = 1 - \frac{2\theta^2}{2^p + 1} + o(\theta^2), \quad b = \frac{2\theta}{2^p + 1} + o(\theta),$$

$$c^2 = a^2 + b^2 = 1 - \frac{2^{p+2}\theta^2}{(2^p + 1)^2} + o(\theta^2)$$

so that

$$c = 1 - \frac{2^{p+1}\theta^2}{(2^p + 1)^2} + o(\theta^2), \quad \frac{1}{c} = 1 + \frac{2^{p+1}\theta^2}{(2^p + 1)^2} + o(\theta^2)$$

and

$$(1 + c)^{1/p} = 2^{1/p} \left( 1 - \frac{2^p \theta^2}{p(2^p + 1)^2} \right) + o(\theta^2), \quad (1 - c)^{1/p} = o(\theta^2)$$

thanks to  $p \in (0, 1)$ . Therefore, the  $(1, 1)$  entry of  $((A^p + B_\theta^p)/2)^{1/p}$  is

$$\begin{aligned}\alpha_p^{(1,1)} &= \left(\frac{2^p + 1}{4}\right)^{1/p} \left\{ 2^{\frac{1}{p}-1} \left(1 - \frac{2^p \theta^2}{p(2^p + 1)^2}\right) \right. \\ &\quad \left. + 2^{\frac{1}{p}-1} \left(1 - \frac{2^p \theta^2}{p(2^p + 1)^2}\right) \left(1 + \frac{2^{p+1} \theta^2}{(2^p + 1)^2}\right) \left(1 - \frac{2\theta^2}{2^p + 1}\right) \right\} + o(\theta^2) \\ &= \frac{(2^p + 1)^{1/p}}{2^{\frac{1}{p}+1}} \left\{ 2 - \frac{2^{p+1} \theta^2}{p(2^p + 1)^2} + \frac{2^{p+1} \theta^2}{(2^p + 1)^2} - \frac{2\theta^2}{2^p + 1} \right\} + o(\theta^2) \\ &= \frac{(2^p + 1)^{1/p}}{2^{1/p}} \left(1 - \frac{2^p + p}{p(2^p + 1)^2} \theta^2\right) + o(\theta^2).\end{aligned}$$

The  $(2, 2)$ -entry of  $((A^p + B_\theta^p)/2)^{1/p}$  is

$$\begin{aligned}\alpha_p^{(2,2)} &= \left(\frac{2^p + 1}{4}\right)^{1/p} \left\{ 2^{\frac{1}{p}-1} \left(1 - \frac{2^p \theta^2}{p(2^p + 1)^2}\right) \right. \\ &\quad \left. - 2^{\frac{1}{p}-1} \left(1 + \frac{2^{p+1} \theta^2}{(2^p + 1)^2}\right) \left(1 - \frac{2^p \theta^2}{p(2^p + 1)^2}\right) \left(1 - \frac{2\theta^2}{2^p + 1}\right) \right\} + o(\theta^2) \\ &= \frac{(2^p + 1)^{1/p}}{2^{\frac{1}{p}+1}} \left\{ -\frac{2^{p+1} \theta^2}{(2^p + 1)^2} + \frac{2\theta^2}{2^p + 1} \right\} + o(\theta^2) \\ &= \frac{(2^p + 1)^{\frac{1}{p}-2}}{2^{1/p}} \theta^2 + o(\theta^2).\end{aligned}$$

The  $(1, 2)$ -entry of  $((A^p + B_\theta^p)/2)^{1/p}$  is

$$\begin{aligned}\alpha_p^{(1,2)} &= \left(\frac{2^p + 1}{4}\right)^{1/p} 2^{\frac{1}{p}-1} \left(1 + \frac{2^{p+1} \theta^2}{(2^p + 1)^2}\right) \left(1 - \frac{2^p \theta^2}{p(2^p + 1)^2}\right) \frac{2\theta}{2^p + 1} + o(\theta^2) \\ &= \frac{(2^p + 1)^{\frac{1}{p}-1}}{2^{1/p}} \theta + o(\theta^2).\end{aligned}$$

By the above estimate for  $((A^p + B_\theta^p)/2)^{1/p}$  and the same for  $((A^q + B_\theta^q)/2)^{1/q}$  we obtain

$$\begin{aligned}\det &\left\{ \left(\frac{A^q + B_\theta^q}{2}\right)^{1/q} - \left(\frac{A^p + B_\theta^p}{2}\right)^{1/p} \right\} \\ &= \{\alpha_q^{(1,1)} - \alpha_p^{(1,1)}\} \{\alpha_q^{(2,2)} - \alpha_p^{(2,2)}\} - \{\alpha_q^{(1,2)} - \alpha_p^{(1,2)}\}^2 \\ &= \left\{ \frac{(2^q + 1)^{1/q}}{2^{1/q}} - \frac{(2^p + 1)^{1/p}}{2^{1/p}} \right\} \left\{ \frac{(2^q + 1)^{\frac{1}{q}-2}}{2^{1/q}} - \frac{(2^p + 1)^{\frac{1}{p}-2}}{2^{1/p}} \right\} \theta^2 \\ &\quad - \left\{ \frac{(2^q + 1)^{\frac{1}{q}-1}}{2^{1/q}} - \frac{(2^p + 1)^{\frac{1}{p}-1}}{2^{1/p}} \right\}^2 \theta^2 + o(\theta^2) \\ &= \left\{ -\frac{(2^p + 1)^{\frac{1}{p}-2} (2^q + 1)^{1/q}}{2^{\frac{1}{p}+\frac{1}{q}}} - \frac{(2^p + 1)^{1/p} (2^q + 1)^{\frac{1}{q}-1}}{2^{\frac{1}{p}+\frac{1}{q}}} \right\}\end{aligned}$$

$$\begin{aligned}
& + \frac{2(2^p + 1)^{\frac{1}{p}-1}(2^q + 1)^{\frac{1}{q}-1}}{2^{\frac{1}{p}+\frac{1}{q}}} \Big\} \theta^2 + o(\theta^2) \\
& = - \left( \frac{2^p + 1}{2} \right)^{1/p} \left( \frac{2^q + 1}{2} \right)^{1/q} \left( \frac{1}{2^p + 1} - \frac{1}{2^q + 1} \right)^2 \theta^2 + o(\theta^2) \quad \text{as } \theta \rightarrow 0.
\end{aligned}$$

□

Now, let  $0 < p < q < 1$ . Suppose that  $((X^p + Y^p)/2)^{1/p} \leq ((X^q + Y^q)/2)^{1/q}$  for all  $X, Y \in \mathbb{P}_2$ . By continuity this holds for all  $2 \times 2$  positive semidefinite  $X, Y$  too so that

$$\left( \frac{A^p + B_\theta^p}{2} \right)^{1/p} \leq \left( \frac{A^q + B_\theta^q}{2} \right)^{1/q}$$

holds for any  $\theta > 0$ . Then, by Lemma 3.3 we must have

$$- \left( \frac{1}{2^p + 1} - \frac{1}{2^q + 1} \right)^2 \geq 0,$$

which implies that  $p = q$ , a contradiction.

### 3.4 Proof of Theorem 2.4

To prove Theorem 2.4, we may, in the same way as above for Theorem 2.3, provide counterexamples for the three cases of Sections 3.1–3.3. This can easily be done by using the same examples as above.

*Case 3.1.* Define a unital CP map (i.e., completely positive linear map)  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$  by

$$\Phi(Z) := \frac{Z[1, 2] + U_\theta Z[1, 3] U_\theta^*}{2}$$

for  $Z \in \mathbb{M}_3$ , where  $Z[i, j]$  denotes the principal submatrix of  $Z$  on rows and columns  $i$  and  $j$ , and

$$U_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

For a diagonal matrix  $Z := \text{diag}(1, x, y)$ , since

$$\Phi(Z^p)^{1/p} = \left( \frac{A^p + B_\theta^p}{2} \right)^{1/p}$$

with  $A$  and  $B_\theta$  in Section 3.1, we have a counterexample for this case in the same way as in Section 3.1.

*Case 3.2.* By the same  $\Phi$  and  $Z$  as in Case 3.1 we have a counterexample as in Section 3.2 since  $\Phi(Z^p)^{1/p}$  for  $p = 0$  is  $\exp((\log A + \log B_\theta)/2)$ .



*Case 3.3.* Define a unital CP map  $\Phi : \mathbb{M}_3 \rightarrow \mathbb{M}_2$  by

$$\Phi(Z) := \frac{Z[1, 3] + U_\theta Z[2, 3]U_\theta^*}{2},$$

where  $U_\theta$  is as in Case 3.1. For a diagonal matrix  $Z := \text{diag}(2, 1, 0)$ , since  $\Phi(Z^p)^{1/p} = ((A^p + B_\theta^p)/2)^{1/p}$  with  $A$  and  $B_\theta$  in Section 3.3, we have a counterexample for this case.

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